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Topological B-model on weighted projective spaces and self-dual models in four dimensions

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Abstract: It was recently shown by Witten on the basis of several examples that the topological B-model whose target space is a Calabi-Yau (CY) supermanifold is equivalent to holomorphic Chern-Simons (hCS) theory on the same supermanifold. Moreover, for the supertwistor space $\mathbb{C}P^{3|4}$ as target space, it has been demonstrated that hCS theory on $\mathbb{C}P^{3|4}$ is equivalent to self-dual $\mathcal{N}=4$ super Yang-Mills (SYM) theory in four dimensions. We consider as target spaces for the B-model the weighted projective spaces $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with two fermionic coordinates of weight p and q, respectively — which are CY supermanifolds for $p+q=4$ — and discuss hCS theory on them. By using twistor techniques, we obtain certain field theories in four dimensions which are equivalent to hCS theory. These theories turn out to be self-dual truncations of $\mathcal{N}=4$ SYM theory or of its twisted (topological) version.

Keywords: Chern-Simons Theories, Superspaces.

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1. Introduction and results

Recently, Witten has shown [1] that there is a one-to-one correspondence between the moduli spaces of holomorphic Chern-Simons (hCS) theory [2] on the Calabi-Yau (CY) supermanifold $\mathbb{C}P^{3|4} \setminus \mathbb{C}P^{1|4}$ and of self-dual $\mathcal{N}=4$ super Yang-Mills (SYM) theory on $\mathbb{R}^4$ with a metric of signature $(++++)$ or $(++--)$. It was also shown that hCS theory on $\mathbb{C}P^{3|4}$ is related with the B-type open topological string (the B-model) having $\mathbb{C}P^{3|4}$ as target space (for related works see [3]–[9]). Therefore, string theory considerations can help to clarify properties of quantum $\mathcal{N}=4$ SYM theory (see e.g. [10]–[13]).[1]

Recall that the fermionic coordinates η_i $(i=1,\ldots,4)$ of the supermanifold $\mathbb{C}P^{3|4} = W\mathbb{C}P^{3|4}(1,1,1,1|1,1,1,1)$ take values in the holomorphic line bundle $\mathcal{O}(1)$. As the product $\prod_{i=1}^4 \mathrm{d}\eta_i$ takes values in $\mathcal{O}(-4)$ and the holomorphic measure on $\mathbb{C}P^3$ is $\mathcal{O}(4)$ valued, the full holomorphic measure on $\mathbb{C}P^{3|4}$ is globally defined and nowhere vanishing. Thus, the supertwistor space $\mathbb{C}P^{3|4}$ is a CY supermanifold [1]. In this note, we will focus on the weighted projective spaces $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with (p,q) equal to $(1,3)$, $(2,2)$ and $(4,0)$. Since $p+q=4$ for these spaces, they are CY supermanifolds due to the above reasoning. We describe hCS theory on open subsets of these weighted projective spaces and derive (equivalent) field theories on $\mathbb{R}^4$ from it. We show that these models are related with different self-dual truncations of either $\mathcal{N}=4$ SYM theory or its twisted (topological) version.

[1]For properties of gravity amplitudes see e.g. [14, 8].

2. Twistor geometry

Local coordinates. Our starting point is the complex projective space $\mathbb{C}P^3$ (twistor space) with homogeneous coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$. These coordinates are subject to the equivalence relation $(\omega^\alpha, \lambda_{\dot\alpha}) \sim (t\omega^\alpha, t\lambda_{\dot\alpha})$ with $t \in \mathbb{C}^*$. Here and in the following, $\alpha, \beta, \ldots = 1, 2$ and $\dot\alpha, \dot\beta, \ldots = 1, 2$ are spinor indices. Consider now the space $\mathcal{P}^3 = \mathbb{C}P^3 \setminus \mathbb{C}P^1$ in which the $\lambda_{\dot\alpha}$ are not simultaneously zero. This space can be covered by two coordinate patches, say $\mathcal{U}_+$ $(\lambda_{\dot 1} \neq 0)$ and $\mathcal{U}_-$ $(\lambda_{\dot 2} \neq 0)$, with coordinates

$$\begin{aligned} z_+^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 1}}\,, \qquad & z_+^3 &= \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}} =: \lambda_+ \qquad \text{on} \qquad \mathcal{U}_+\,, \\ z_-^\alpha &= \frac{\omega^\alpha}{\lambda_{\dot 2}}\,, \qquad & z_-^3 &= \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}} =: \lambda_- \qquad \text{on} \qquad \mathcal{U}_-\,. \end{aligned} \tag{2.1}$$

On the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$, they are related by

$$z_+^\alpha = \lambda_+ z_-^\alpha \qquad \text{and} \qquad \lambda_+ = \lambda_-^{-1}\,. \tag{2.2}$$

It follows from (2.1) and (2.2) that $\mathcal{P}^3$ coincides with the total space of the rank 2 holomorphic vector bundle[2] $\mathbb{C}^2 \otimes \mathcal{O}(1)$ over the Riemann sphere $\mathbb{C}P^1$. We denote the covering of the latter by $U_\pm = \mathcal{U}_\pm \cap \mathbb{C}P^1$ with coordinates $\lambda_\pm$ on $U_\pm$.

Real structure. As we are eventually interested in real fields, we need to introduce a real structure on $\mathcal{P}^3$. This can be done by defining the anti-linear transformations

$$\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \mapsto \begin{pmatrix} -\bar\omega^2 \\ \bar\omega^1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \lambda_{\dot 1} \\ \lambda_{\dot 2} \end{pmatrix} \mapsto \begin{pmatrix} -\bar\lambda_{\dot 2} \\ \bar\lambda_{\dot 1} \end{pmatrix} \tag{2.3}$$

which induce an antiholomorphic involution (real structure) τ on $\mathcal{P}^3$. On the coordinates (2.1), it acts as follows:

$$\tau(z_\pm^1, z_\pm^2, z_\pm^3) = \left(\pm \frac{\bar z_\pm^2}{\bar z_\pm^3}, \mp \frac{\bar z_\pm^1}{\bar z_\pm^3}, -\frac{1}{\bar z_\pm^3} \right). \tag{2.4}$$

It is not difficult to see that τ has no fixed points in $\mathcal{P}^3$ but does leave invariant projective lines $\mathbb{C}P^1$ joining the points p and $\tau(p)$ for any $p \in \mathcal{P}^3$. For two other possible real structures on $\mathcal{P}^3$, see e.g. [15].

Real rational curves. Holomorphic sections of the vector bundle $\mathcal{P}^3 \to \mathbb{C}P^1$ are rational degree one curves $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$ defined by the equations

$$z_\pm^\alpha = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}^\pm \qquad \text{for} \quad (\lambda_{\dot\alpha}^+) = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad \lambda_{\dot\alpha}^- = \lambda_+^{-1}\lambda_{\dot\alpha}^+ \quad \text{with} \quad \lambda_\pm \in U_\pm \tag{2.5}$$

and parametrized by the moduli $(x^{\alpha\dot\alpha}) \in \mathbb{C}^4$. Thus, the complexified spacetime $\mathbb{C}^4$ can be identified with the moduli space of holomorphic sections of the bundle $\mathcal{P}^3 \to \mathbb{C}P^1$.

[2]Recall that the holomorphic line bundle $\mathcal{O}(m)$ over $\mathbb{C}P^1$ is defined by the transition function λ_+^m.

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The action of the map τ on the $x^{\alpha\dot\alpha}$ is given by

$$\tau\begin{pmatrix} x^{1\dot1} & x^{1\dot2} \\ x^{2\dot1} & x^{2\dot2}\end{pmatrix} = \begin{pmatrix} \bar x^{2\dot2} & -\bar x^{2\dot1} \\ -\bar x^{1\dot2} & \bar x^{1\dot1}\end{pmatrix}, \tag{2.6}$$

where the overbar denotes complex conjugation. The real subspace $\mathbb{R}^4$ of $\mathbb{C}^4$ invariant under τ is defined by the equations

$$x^{2\dot2} = \bar x^{1\dot1} =: x^1 - \mathrm{i}x^2 \qquad \text{and} \qquad x^{2\dot1} = -\bar x^{1\dot2} =: x^3 - \mathrm{i}x^4 \tag{2.7}$$

and parametrized by the real coordinates $x = (x^\mu) \in \mathbb{R}^4$ with $\mu = 1,\ldots,4$. The action of τ on arbitrary functions $f(x,\lambda_\pm,\bar\lambda_\pm)$ is then immediate. Hence, one may impose the condition of invariance under τ on sections of the form (2.5) and realize that they are real if the conditions (2.7) hold true.

Metric and signature. On the space $\mathbb{R}^4$ of τ-real holomorphic curves $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}^3$, one can introduce the metric

$$\mathrm{d}s^2 = \det(\mathrm{d}x^{\alpha\dot\alpha}) = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,, \tag{2.8}$$

which is euclidean, $(g_{\mu\nu}) = \mathrm{diag}(+1,+1,+1,+1)$, due to (2.7). Other real structures on $\mathcal{P}^3$ produce a metric of signature $(++--)$ on $\mathbb{R}^4$ (see e.g. [15]).

Note that topologically, $\mathcal{P}^3$ is the direct product

$$\mathbb{R}^4 \times \mathbb{C}P^1 \cong \mathcal{P}^3 \tag{2.9}$$

with coordinates $(x^\mu,\lambda_\pm)$. Therefore, on the patches $\mathcal{U}_\pm$ covering $\mathcal{P}^3$ one can use either the complex coordinates (2.1) or coordinates $(x^\mu,\lambda_\pm)$. In fact, the formulas (2.5) together with their complex conjugates define a diffeomorphism of $\mathbb{R}^4\times\mathbb{C}P^1$ onto $\mathcal{P}^3$. Its inverse is given by

$$x^{1\dot1} = \frac{z^1_+ + z^3_+\bar z^2_+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_- z^1_- + \bar z^2_-}{1+z^3_-\bar z^3_-}\,, \qquad x^{1\dot2} = -\frac{\bar z^2_+ - \bar z^3_+ z^1_+}{1+z^3_+\bar z^3_+} = -\frac{z^3_-\bar z^2_- - z^1_-}{1+z^3_-\bar z^3_-}\,, \tag{2.10}$$

together with $\lambda_\pm = z^3_\pm$ and (2.7) for $x^{2\dot2}$ and $x^{2\dot1}$.

Vector fields. On the complex manifold $\mathcal{P}^3$, we have a natural basis $(\partial/\partial\bar z^\alpha_\pm, \partial/\partial\bar z^3_\pm)$ for antiholomorphic vector fields. These are sections of the bundles[3] $\overline{\mathcal{O}}(-1)$ and $\overline{\mathcal{O}}(-2)$, respectively. Using the formulas (2.10), we can express them in terms of the coordinates $(x^{\alpha\dot\alpha},\lambda_\pm)$ according to

$$\begin{aligned} \partial_{\bar z^1_\pm} &= -\gamma_\pm\lambda^{\dot\alpha}_\pm\partial_{2\dot\alpha} =: -\gamma_\pm\bar V^\pm_2\,, \qquad \partial_{\bar z^2_\pm} = \gamma_\pm\lambda^{\dot\alpha}_\pm\partial_{1\dot\alpha} =: \gamma_\pm\bar V^\pm_1\,, \\ \partial_{\bar z^3_\pm} &= \partial_{\bar\lambda_\pm} + \gamma_\pm x^{\alpha\dot1}\bar V^\pm_\alpha =: \bar V^\pm_3 + \gamma_\pm x^{\alpha\dot1}\bar V^\pm_\alpha\,, \end{aligned} \tag{2.11}$$

where

$$\lambda^{\dot\alpha}_\pm = \epsilon^{\dot\alpha\dot\beta}\lambda^\pm_{\dot\beta} \qquad \text{with} \quad \epsilon^{\dot1\dot2} = 1 \quad \text{and} \quad \gamma_\pm = (1+\lambda_\pm\bar\lambda_\pm)^{-1}\,. \tag{2.12}$$

[3]Here, $\overline{\mathcal{O}}(m)$ denotes the bundle which is complex conjugate to $\mathcal{O}(m)$.

We use $\epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha}$. Thus, the vector fields

$$\bar V^{\pm}_{\alpha} = \lambda^{\dot\alpha}_{\pm}\partial_{\alpha\dot\alpha} \qquad \text{and} \qquad \bar V^{\pm}_{3} \;=\; \partial_{\bar\lambda_{\pm}} \tag{2.13}$$

form a basis of vector fields of type $(0,1)$ on $\mathcal{U}_{\pm} \subset \mathcal{P}^3$ in the coordinates $(x^{\alpha\dot\alpha}, \lambda_{\pm})$. For more details on the geometry of the twistor space $\mathcal{P}^3$, see e.g. [16] and references therein.

3. Holomorphic Chern-Simons theory on $W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$

Weighted projective space $W^{3|2}(p,q)$. Let us consider the weighted projective space $W^{3|2}(p,q) = W\mathbb{C}P^{3|2}(1,1,1,1|p,q)$ with four homogeneous bosonic coordinates $(\omega^\alpha, \lambda_{\dot\alpha})$ of weight one and two fermionic coordinates $(\rho_i) = (\rho_1,\rho_2)$ of weight p and q, respectively. Here, $(\omega^\alpha, \lambda_{\dot\alpha})$ are the homogeneous coordinates on $\mathbb{C}P^3$. Together, the coordinates $(\omega^\alpha, \lambda_{\dot\alpha}, \rho_i)$ are subject to the equivalence relation $(\omega^\alpha, \lambda_{\dot\alpha}, \rho_1, \rho_2) \sim (t\omega^\alpha, t\lambda_{\dot\alpha}, t^p\rho_1, t^q\rho_2)$ for all $t \in \mathbb{C}^*$. We assume that the sum of fermionic weights is equal to the sum of bosonic weights, i.e. $p+q=4$. In this case, $W^{3|2}(p,q)$ is a CY supermanifold and one may consider the topological B-model of $W^{3|2}(p,q)$ [1]. The same arguments as given in [1] then yield hCS theory on this space.

Action of hCS theory. Let $\mathcal{E}$ be a trivial rank n complex vector bundle over $W^{3|2}(p,q)$ and $\mathcal{A}$ a connection one-form on $\mathcal{E}$. Consider a subspace $\mathcal{Y}$ of $W^{3|2}(p,q)$ which is parametrized by the holomorphic and antiholomorphic bosonic coordinates as well as the holomorphic fermionic coordinates.[4] The action of hCS theory then reads as [1]

$$S_{\text{hCS}} = \int_{\mathcal{Y}} \Omega \wedge \operatorname{tr}\left(\mathcal{A}^{0,1} \wedge \bar\partial \mathcal{A}^{0,1} + \frac{2}{3}\mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1}\right), \tag{3.1}$$

where Ω is a nowhere vanishing holomorphic volume form and $\mathcal{A}^{0,1}$ denotes the $(0,1)$-component of $\mathcal{A}$. It is assumed that $\mathcal{A}^{0,1}$ contains no antiholomorphic fermionic components and depends only on ρ_i as well as on $(\omega^\alpha, \lambda_{\dot\alpha})$ and $(\bar\omega^\alpha, \bar\lambda_{\dot\alpha})$. The action (3.1) leads to the field equations

$$\bar\partial \mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0\,. \tag{3.2}$$

Note that for constructing solutions to these equations, one can use a generalization of the Riemann-Hilbert problem based on the equivalence of the Čech and Dolbeault descriptions of holomorphic bundles [17].

Local coordinates. Consider now an open subset of $W^{3|2}(p,q)$ defined as $\mathcal{WP}^{3|2}(p,q) = W\mathbb{C}P^{3|2}(1,1,1,1|p,q) \setminus W\mathbb{C}P^{1|2}(1,1|p,q)$. It can be covered by two patches, which we denote by $\hat{\mathcal{U}}_{\pm}$. Since the body of the CY supermanifold $\mathcal{WP}^{3|2}(p,q)$ is the twistor space $\mathcal{P}^3$, we may take the coordinates given in (2.1) as bosonic coordinates on $\hat{\mathcal{U}}_{\pm}$ and as fermionic ones

$$\rho^+_1 = \frac{\rho_1}{\lambda^p_{\dot 1}}\,, \quad \rho^+_2 = \frac{\rho_2}{\lambda^q_{\dot 1}} \quad \text{on} \quad \hat{\mathcal{U}}_+ \qquad \text{and} \qquad \rho^-_1 = \frac{\rho_1}{\lambda^p_{\dot 2}}\,, \quad \rho^-_2 = \frac{\rho_2}{\lambda^q_{\dot 2}} \quad \text{on} \quad \hat{\mathcal{U}}_-\,, \tag{3.3}$$

[4] Recall that $\mathcal{Y}$ is the world-volume of the D-branes which are not quite space-filling [1].

which are related by $\rho_1^+ = \lambda_+^p \rho_1^-$ and $\rho_2^+ = \lambda_+^q \rho_2^-$ on the intersection $\hat{\mathcal{U}}_+ \cap \hat{\mathcal{U}}_-$. Note that as bosonic coordinates on $\mathcal{WP}^{3|2}(p,q)$, we can use either $(z_\pm^\alpha, z_\pm^3)$ or $(x^{\alpha\dot\alpha}, \lambda_\pm)$ and similarly for the vector fields (2.11) and (2.13). The vector fields (2.13) together with $\partial/\partial\bar\rho_i$ form a basis of vector fields of type $(0,1)$ on $\mathcal{WP}^{3|2}(p,q)$.

HCS theory on $\mathcal{WP}^{3|2}(p,q)$. Having given all the ingredients, we may now consider hCS theory on $\mathcal{WP}^{3|2}$ with $p+q=4$. The equations of motion (3.2) on the patches $\hat{\mathcal{U}}_\pm$ of $\mathcal{WP}^{3|2}$ read

$$\begin{aligned} \bar V_\alpha^\pm \mathcal{A}_\beta^\pm - \bar V_\beta^\pm \mathcal{A}_\alpha^\pm + [\mathcal{A}_\alpha^\pm, \mathcal{A}_\beta^\pm] &= 0\,, \\ \bar V_\alpha^\pm \mathcal{A}_{\bar\lambda_\pm} - \partial_{\bar\lambda_\pm} \mathcal{A}_\alpha^\pm + [\mathcal{A}_\alpha^\pm, \mathcal{A}_{\bar\lambda_\pm}] &= 0\,, \end{aligned} \tag{3.4}$$

where

$$\mathcal{A}_\alpha^\pm := \bar V_\alpha^\pm \lrcorner \mathcal{A}^{0,1} \qquad \text{and} \qquad \mathcal{A}_{\bar\lambda_\pm} := \partial_{\bar\lambda_\pm} \lrcorner \mathcal{A}^{0,1}\,. \tag{3.5}$$

Here, "$\lrcorner$" denotes the interior product of vector fields with differential forms. As usual in the twistor approach, we shall assume that there exists a gauge in which the components $\mathcal{A}_{\bar\lambda_\pm}$ are zero. This corresponds to the holomorphic triviality on any $\mathbb{C}P^1_{x,\rho} \hookrightarrow \mathcal{WP}^{3|2}(p,q)$ of the holomorphic vector bundle which is associated to any solution of hCS theory.

Note that generically, the gauge field $\mathcal{A}^{0,1}$ on $\mathcal{WP}^{3|2}(p,q)$ can be expanded in terms of the odd coordinates ρ_i according to

$$\mathcal{A}^{0,1} = A + \rho_i \psi^i + \rho_1\rho_2 G\,, \tag{3.6}$$

Linearizing the equations (3.2) around the trivial solution $\mathcal{A}^{0,1} = 0$, we see that $\bar\partial \mathcal{A}^{0,1} = 0$ or equivalently $\bar\partial A = 0$, $\bar\partial\psi^i = 0$ and $\bar\partial G = 0$. Since $\mathcal{A}^{0,1}$ is $\mathcal{O}(0)$-valued and ρ_1 and ρ_2 take values in $\mathcal{O}(p)$ and $\mathcal{O}(q)$, respectively, the fields A, ψ^1, ψ^2 and G are $(0,1)$-forms on $\mathcal{P}^3$ with values in the bundles $\mathcal{O}(0)$, $\mathcal{O}(-p)$, $\mathcal{O}(-q)$ and $\mathcal{O}(-4)$. Therefore, each of these fields determines an element of the sheaf cohomology group $H^1(\mathcal{P}^3, \mathcal{O}(2s-2))$, where $s=1$ for A, $s = 1 - \frac12 p$ for ψ^1, $s = 1-\frac12 q$ for ψ^2 and $s=-1$ for G. According to the Penrose transform [18], elements of these cohomology groups correspond to solutions of the field equations for massless fields of helicity s on the space $\mathbb{R}^4$; the latter being parametrized by $x^{\alpha\dot\alpha}$. In the next section, we describe this correspondence beyond linearized level.

4. Field models on $\mathbb{R}^4$ equivalent to hCS theory on $\mathcal{WP}^{3|2}(p,q)$

From now on, we shall work in the patch $\hat{\mathcal{U}}_+$ of $\mathcal{WP}^{3|2}(p,q)$ and for notational simplicity we omit the subscript and the superscript "+" in all expressions except when confusion may arise.

HCS theory on $\mathcal{WP}^{3|2}(1,3)$. Let us consider the case $p=1$ and $q=3$, where the fermionic coordinates ρ_1 and ρ_2 take values in the bundles $\mathcal{O}(1)$ and $\mathcal{O}(3)$, respectively. From (2.13) and (3.5) one concludes that the components $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_{\bar\lambda}$ of the $(0,1)$-form $\mathcal{A}^{0,1}$ take values in the bundles $\mathcal{O}(1)$, $\mathcal{O}(1)$ and $\overline{\mathcal{O}}(-2)$. This fixes the dependence of $\mathcal{A}_\alpha$ and $\mathcal{A}_{\bar\lambda}$ on λ and $\bar\lambda$ up to gauge transformations. Namely, we obtain

$$\mathcal{A}_\alpha = \lambda^{\dot\alpha} A_{\alpha\dot\alpha} + \rho_1\chi_\alpha + \frac{1}{2!\sqrt3}\rho_2\gamma^2\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\widetilde\chi_{\alpha\dot\alpha\dot\beta} + \frac{1}{3!}\rho_1\rho_2\gamma^3\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\hat\lambda^{\dot\gamma}G_{\alpha\dot\alpha\dot\beta\dot\gamma}\,,$$

$$\mathcal{A}_{\bar\lambda} = \frac{1}{\sqrt{3}}\rho_2\gamma^3\hat\lambda^{\dot\alpha}\widetilde\chi_{\dot\alpha} + \frac{1}{2!}\rho_1\rho_2\gamma^4\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}G_{\dot\alpha\dot\beta}\,, \tag{4.1}$$

where

$$\hat\lambda^{\dot\alpha} := \tau(\lambda^{\dot\alpha}) = \bar\lambda_{\dot\alpha}\,. \tag{4.2}$$

In (4.1), all the fields $A_{\alpha\dot\alpha}$, χ_α, $\widetilde\chi_{\dot\alpha}$, $G_{\dot\alpha\dot\beta}$, $\widetilde\chi_{\alpha\dot\alpha\dot\beta}$ and $G_{\alpha\dot\alpha\dot\beta\dot\gamma}$ depend only on $(x^{\alpha\dot\alpha}) \in \mathbb{R}^4$.

Substituting (4.1) into (3.4), we get the equations

$$\widetilde\chi_{\alpha\dot\alpha\dot\beta} = -\nabla_{\alpha(\dot\alpha}\widetilde\chi_{\dot\beta)} \qquad \text{and} \qquad G_{\alpha\dot\alpha\dot\beta\dot\gamma} = -\nabla_{\alpha(\dot\alpha}G_{\dot\beta\dot\gamma)} \tag{4.3}$$

showing that $\widetilde\chi_{\alpha\dot\alpha\dot\beta}$ and $G_{\alpha\dot\alpha\dot\beta\dot\gamma}$ are composite fields describing no independent degrees of freedom. Here, parentheses denote normalized symmetrization with respect to the enclosed indices and $\nabla_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha},\,\cdot\,]$. The remaining equations are given by

$$f_{\dot\alpha\dot\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}(\partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha}, A_{\beta\dot\beta}]) = 0\,, \tag{4.4a}$$

$$\nabla_{\alpha\dot\alpha}\chi^\alpha = 0\,, \tag{4.4b}$$

$$\nabla_{\alpha\dot\alpha}\widetilde\chi^{\dot\alpha} = 0\,, \tag{4.4c}$$

$$\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}G_{\dot\beta\dot\gamma} - \{\chi_\alpha, \widetilde\chi_{\dot\gamma}\} = 0\,. \tag{4.4d}$$

Due to (4.4) and the Bianchi identities, all other equations following from (3.4) are automatically satisfied.

The equations (4.4) are the equations of motion for the action functional

$$S = \int \mathrm{d}^4x\,\mathrm{tr}\left\{G^{\dot\alpha\dot\beta}f_{\dot\alpha\dot\beta} + \widetilde\chi^{\dot\alpha}\nabla_{\alpha\dot\alpha}\chi^\alpha\right\} \tag{4.5}$$

which can be obtained from (3.1) by integration over the fermionic coordinates and over the sphere $\mathbb{C}P^1$. Note that this action has an obvious supersymmetry. The transformation laws are given by

$$\begin{aligned} \delta_\xi A_{\alpha\dot\alpha} &= \xi_{\dot\alpha}\chi_\alpha &\qquad \text{and} \qquad \delta_\xi G_{\dot\alpha\dot\beta} &= -\epsilon^{\alpha\beta}\xi_{(\dot\alpha}\nabla_{\alpha\dot\beta)}\chi_\beta\,,\\ \delta_\xi\chi_\alpha &= 0 &\qquad \text{and} \qquad \delta_\xi\widetilde\chi_{\dot\alpha} &= -\xi^{\dot\beta}(G_{\dot\alpha\dot\beta} + f_{\dot\alpha\dot\beta})\,,\end{aligned}$$

where $\xi_{\dot\alpha}$ is a constant (anticommuting) spinor. The action (4.5) describes a truncation of the self-dual $\mathcal{N}=4$ SYM model [19] for which all the scalars and three of the dotted and three of the undotted fermions are put to zero.

HCS theory on $\mathcal{WP}^{3|2}(2,2)$. Now we consider the case $p=q=2$, i.e. the fermionic coordinates ρ_i take values in the line bundle $\mathcal{O}(2)$. The equations of motion of hCS theory on $\mathcal{WP}^{3|2}(2,2)$ have the same form (3.4). Again, the functional dependence on λ and $\bar\lambda$ is fixed up to gauge transformations by the geometry of $\mathcal{WP}^{3|2}(2,2)$. Namely, this dependence has the form

$$\begin{aligned} \mathcal{A}_\alpha &= \lambda^{\dot\alpha}A_{\alpha\dot\alpha} + \frac{1}{2!}\rho_i\gamma\hat\lambda^{\dot\alpha}\phi^i_{\alpha\dot\alpha} + \frac{1}{3!}\rho_1\rho_2\gamma^3\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\hat\lambda^{\dot\gamma}G_{\alpha\dot\alpha\dot\beta\dot\gamma}\,,\\ \mathcal{A}_{\bar\lambda} &= \frac{1}{2!}\rho_i\gamma^2\phi^i + \frac{1}{2!}\rho_1\rho_2\gamma^4\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}G_{\dot\alpha\dot\beta}\,, \end{aligned} \tag{4.7}$$

where again the coefficient fields do only depend on $(x^{\alpha\dot\alpha}) \in \mathbb{R}^4$ and $\lambda^{\dot\alpha}$, γ and $\hat\lambda^{\dot\alpha}$ are given in (2.12) and (4.2), respectively.

Substituting (4.7) into (3.4), we obtain the following equations:

$$\phi^i_{\alpha\dot\alpha} = -\nabla_{\alpha\dot\alpha}\phi^i \qquad \text{and} \qquad G_{\alpha\dot\alpha\dot\beta\dot\gamma} = -\nabla_{\alpha(\dot\alpha}G_{\dot\beta\dot\gamma)}\,. \tag{4.8}$$

The remaining nontrivial equations read

$$f_{\dot\alpha\dot\beta} = 0\,, \tag{4.9a}$$

$$\nabla_{\alpha\dot\alpha}\nabla^{\alpha\dot\alpha}\phi^i = 0\,, \tag{4.9b}$$

$$\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}G_{\dot\beta\dot\gamma} - \frac{3}{4}\epsilon_{ij}\{\phi^i, \nabla_{\alpha\dot\gamma}\phi^j\} = 0\,. \tag{4.9c}$$

The associated action functional is given by

$$S = \int \mathrm{d}^4x \,\mathrm{tr}\left\{G^{\dot\alpha\dot\beta}f_{\dot\alpha\dot\beta} + \frac{3}{8}\epsilon_{ij}\phi^i\nabla_{\alpha\dot\alpha}\nabla^{\alpha\dot\alpha}\phi^j\right\}, \tag{4.10}$$

and can be obtained from (3.1). Note that formally (4.10) looks as the bosonic truncation of the self-dual $\mathcal{N}=4$ SYM theory [19], i.e. all the spinors and four of the six scalars of self-dual $\mathcal{N}=4$ super SYM theory are put to zero. However, in (4.10) the parity of the scalars ϕ^i is different, as they are Graßmann odd. To understand their nature, note that in the expansions (4.7) we have two Graßmann odd vectors $\phi^i_{\alpha\dot\alpha}$ which satisfy the equations

$$\epsilon^{\alpha\beta}\nabla_{\alpha\dot\alpha}\phi^i_{\beta\dot\beta} = 0\,, \tag{4.11a}$$

$$\nabla^{\alpha\dot\alpha}\phi^i_{\alpha\dot\alpha} = 0 \tag{4.11b}$$

following from (3.4). Solutions to these equations describe tangent vectors $\delta A_{\alpha\dot\alpha}$ (with assigned odd parity) to the solution space of the self-duality equations (4.9a) [20, 21]. However, due to the first equation in (4.8) (which solves (4.11a) and reduces (4.11b) to (4.9b)), the $\phi^i_{\alpha\dot\alpha}$ are projected to zero in the moduli space of solutions to the equations (4.9a). By choosing $\phi^i=0$, we remain with the equations

$$f_{\dot\alpha\dot\beta} = 0 \qquad \text{and} \qquad \nabla_{\alpha\dot\alpha}G^{\dot\alpha\dot\beta} = 0\,, \tag{4.12}$$

which can be obtained from the Lorentz invariant Siegel action [19]

$$S = \int \mathrm{d}^4x \,\mathrm{tr}\{G^{\dot\alpha\dot\beta}f_{\dot\alpha\dot\beta}\}\,, \tag{4.13}$$

describing purely bosonic self-dual Yang-Mills theory.

HCS theory on $\mathcal{WP}^{3|2}(4,0)$. Finally, we want to discuss the case in which the fermionic coordinate ρ_1 has weight four and ρ_2 weight zero, i.e. we consider $\mathcal{WP}^{3|2}(4,0)$. The field equations of hCS theory on $\mathcal{WP}^{3|2}(4,0)$ have the form (3.4), where the vector fields $\bar V^\pm_\alpha$ and $\partial_{\bar\lambda_\pm}$ are given in (2.13). Proceeding as in the previous two subsections, we obtain the following field expansions:

$$\mathcal{A}_\alpha = \lambda^{\dot\alpha}A_{\alpha\dot\alpha} + \frac{1}{3!}\rho_1\gamma^3\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\hat\lambda^{\dot\gamma}\chi_{\alpha\dot\alpha\dot\beta\dot\gamma} + \rho_2\lambda^{\dot\alpha}\psi_{\alpha\dot\alpha} + \frac{1}{3!}\rho_1\rho_2\gamma^3\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\hat\lambda^{\dot\gamma}G_{\alpha\dot\alpha\dot\beta\dot\gamma}\,,$$

$$\mathcal{A}_{\bar{\lambda}} = \frac{1}{2!}\rho_1\gamma^4\hat{\lambda}^{\dot{\alpha}}\hat{\lambda}^{\dot{\beta}}\chi_{\dot{\alpha}\dot{\beta}} + \frac{1}{2!}\rho_1\rho_2\gamma^4\hat{\lambda}^{\dot{\alpha}}\hat{\lambda}^{\dot{\beta}}G_{\dot{\alpha}\dot{\beta}}\,. \tag{4.14}$$

All the Lie-algebra valued fields appearing in the expansions (4.14) depend only on $(x^{\alpha\dot{\alpha}}) \in \mathbb{R}^4$. Note also that $A_{\alpha\dot{\alpha}}$, $G_{\dot{\alpha}\dot{\beta}}$ and $G_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}$ are bosonic while $\psi_{\alpha\dot{\alpha}}$, $\chi_{\dot{\alpha}\dot{\beta}}$ and $\chi_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}$ are fermionic fields.

Substituting (4.14) into (3.4), we obtain

$$\chi_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} = -\nabla_{\alpha(\dot{\alpha}}\chi_{\dot{\beta}\dot{\gamma})} \qquad \text{and} \qquad G_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}} = -\nabla_{\alpha(\dot{\alpha}}G_{\dot{\beta}\dot{\gamma})} - \{\psi_{\alpha(\dot{\alpha}}, \chi_{\dot{\beta}\dot{\gamma})}\}\,. \tag{4.15}$$

The remaining nontrivial equations are

$$f_{\dot{\alpha}\dot{\beta}} = 0\,, \tag{4.16a}$$

$$\epsilon^{\alpha\beta}\nabla_{\alpha(\dot{\alpha}}\psi_{\beta\dot{\beta})} = 0\,, \tag{4.16b}$$

$$\nabla_{\alpha\dot{\alpha}}G^{\dot{\alpha}\dot{\beta}} + \{\psi_{\alpha\dot{\alpha}}, \chi^{\dot{\alpha}\dot{\beta}}\} = 0\,, \tag{4.16c}$$

$$\nabla_{\alpha\dot{\alpha}}\chi^{\dot{\alpha}\dot{\beta}} = 0\,. \tag{4.16d}$$

In this case, the action functional from which these equations arise is

$$S = \int \mathrm{d}^4x\,\mathrm{tr}\left\{G^{\dot{\alpha}\dot{\beta}}f_{\dot{\alpha}\dot{\beta}} + \epsilon^{\alpha\beta}(\nabla_{\alpha\dot{\alpha}}\psi_{\beta\dot{\beta}})\chi^{\dot{\alpha}\dot{\beta}}\right\}. \tag{4.17}$$

This time, the multiplet contains a spacetime vector $\psi_{\alpha\dot{\alpha}}$ and an anti-self-dual two-form $\chi_{\dot{\alpha}\dot{\beta}}$ which are both Graßmann odd. Such fields are well known from topological Yang-Mills theories [20, 21]. In this respect, the model (4.16), (4.17) can be understood as a truncated self-dual sector of these theories.[5]

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A. Signature $(++--)$

For the Kleinian space $\mathbb{R}^{2,2} = (\mathbb{R}^4, g)$ with the metric $g = \mathrm{diag}(+1,+1,-1,-1)$, all the calculations and formulas are similar. What should be changed is the following. Instead of the reality conditions (2.7), one considers

$$x^{2\dot{2}} = \bar{x}^{1\dot{1}} =: x^1 - \mathrm{i}x^2 \qquad \text{and} \qquad x^{2\dot{1}} = \bar{x}^{1\dot{2}} =: x^3 + \mathrm{i}x^4 \tag{A.1}$$

which correspond to the involution (real structure)

$$\hat{\tau}(z^1_\pm, z^2_\pm, z^3_\pm) = \left(\frac{\bar{z}^2_\pm}{\bar{z}^3_\pm}, \frac{\bar{z}^1_\pm}{\bar{z}^3_\pm}, \frac{1}{\bar{z}^3_\pm}\right) \tag{A.2}$$

[5] One may also consider more than two fermionic coordinates in order to enlarge the multiplet. This may lead to other truncations of topological Yang-Mills theories.

on the twistor space $\mathcal{P}^3$. Furthermore, instead of $\mathcal{P}^3$ one should consider the space[6] $\widetilde{\mathcal{P}}^3 = \mathcal{P}^3 \setminus \mathcal{T}^3$, where $\mathcal{T}^3$ is the fixed point set under the involution $\hat{\tau}$. In fact, $\mathcal{T}^3$ is the three-dimensional real manifold $\mathbb{R}P^3 \backslash \mathbb{R}P^1$ fibered over $S^1 \cong \mathbb{R}P^1 \subset \mathbb{C}P^1$. All CY supermanifolds $\mathcal{WP}^{3|2}(p,q)$ considered above are now supermanifolds with $\widetilde{\mathcal{P}}^3$ as body. One should also substitute $\mathbb{C}P^1$ by the two-sheeted hyperboloid $H^2 = \mathbb{C}P^1 \setminus S^1 = H_+ \cup H_-$ and use

$$(\hat{\lambda}_+^{\dot{\alpha}}) = \begin{pmatrix} -1 \\ \bar{\lambda}_+ \end{pmatrix}, \qquad (\hat{\lambda}_-^{\dot{\alpha}}) = \begin{pmatrix} -\bar{\lambda}_- \\ 1 \end{pmatrix}, \qquad \gamma_\pm = \pm(1 - \lambda_\pm \bar{\lambda}_\pm)^{-1} \qquad \text{with} \quad \lambda_\pm \in H_\pm \tag{A.3}$$

instead of $\gamma_\pm$ and $\hat{\lambda}_\pm^{\dot{\alpha}}$ as given in (2.12) and (4.2), respectively. All other formulas, including the equations of motion for hCS theory and the field expansions of $\mathcal{A}^{0,1}$ keep their form. The resulting field equations on $\mathbb{R}^{2,2}$ will only differ by some signs in front of the interaction terms.

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